Performance Bounds for Robust Quadratic Dynamic Matrix Control with End Condition

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Sufficient conditions for robust stability of multivariable quadratic dynamic matrix controllers with an end condition (EQDMC) are developed, and the effect of these conditions on closed-loop performance is examined. Hard and soft constraints on process inputs and outputs, and process modeling uncertainty are present. Modeling uncertainty is quantified in the time domain as upper and lower bounds on the coefficients of a finite pulse response process model. The robust stability conditions that we develop involve the prediction and control horizon lengths, and a set of inequalities that the move-suppression coefficients in the on-line EQDMC objective function must satisfy. These conditions imply that for processes with modeling uncertainty, the move-suppression coefficients could be quite large and quite sensitive to the control horizon length and process modeling errors. This could make the EQDMC controller conservative, thus significantly deteriorating performance. To determine the optimal control horizon length corresponding to the best robust performance of the EQDMC controller for a class of disturbances, follow the proposed EQDMC controller design methodology. To illustrate this methodology simulations on an SISO example and a 2 × 2 subsystem of the Shell Standard Control Problem are presented.

Introduction

Studies on the analysis and synthesis of robust model-predictive control (MPC) have drawn a great deal of interest from researchers in this area. Reasons that make the MPC problem interesting are

- The presence of process input and output constraints, which render the closed-loop nonlinear, even if the process is linear;
- The need for a suitable uncertainty description, so that the robust stability and performance of an MPC closed loop can be studied.

Robust stability and performance conditions for MPC without process input and output constraints are discussed in Morari and Zafiriou (1989). The analysis and synthesis methods presented therein are developed in the *frequency* domain. For linear processes with constraints, but no uncertainty, Zafiriou (1988) provided guidelines for the tuning of a class of MPC. Rawlings and Muske (1993) developed stability conditions for finite-horizon-constrained MPC controllers that guarantee infinite horizon stability. Meadows and Rawlings (1993) extended the ideas in Rawlings and Muske (1993) to constrained nonlinear systems. Genceli and Nikolaou (1993) developed sufficient conditions for robust stability of single-input single-output (SISO) MPC systems with l_1 on-line objective functions, respectively. Zheng and Morari (1993) presented robust stability conditions based on a min-max approach.

In this article we formulate the multivariable quadratic dynamic matrix control (QDMC) problem with an end condition (EQDMC) and present a generalization of the robust stability and performance conditions developed by Genceli and Nikolaou (1993b) for SISO systems with input/output constraints and modeling uncertainty. We focus on the effect of modeling uncertainty on closed-loop *performance*. As in Genceli and Nikolaou (1993), keys to deriving the robust stability and performance conditions are

- Satisfaction, by process inputs, of an end condition in the moving horizon on-line optimization problem
- Characterization of process modeling uncertainty in the *time* domain, as lower and upper bounds on the coefficients of the pulse-response model of the process.

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Controller-tuning parameters that appear in the robust stability and performance conditions include:

- Values of the weights in the control move-suppression terms of the on-line objective function
- The lengths of the prediction and control horizons.

Although frequency domain methods have been proved extremely fertile for robust control of linear systems, they have not been equally successful for constrained (hence nonlinear) MPC. Moreover, the experimental or computational determination of frequency domain uncertainty as an upper bound on the norm of the process model perturbation can be quite demanding. On the other hand, description of modeling uncertainty in the time domain can greatly capitalize on a plethora of experimentally convenient parameter identification methods that use statistics to provide confidence intervals for the parameters of finite pulse response models (see, for example, Box and Jenkins, 1976). While statistical methods provide only soft bounds, hard bounds corresponding to probability "close" to 1 may be established. In addition to the statistical approach, an optimal system identification for control purposes can be performed according to the ideas in Chen et al. (1992), who develop bounds for step and pulse response models as the result of solving a simple linear programming problem.

The robust stability conditions that we develop in the sequel dictate values for the weights in the move-suppression terms of the on-line objective function that increase as modeling uncertainty increases. This results in closed-loop performance that initially improves, as we start from a very short control horizon and expand it, but later deteriorates, as we extend the control horizon to excessively large lengths. One can find an intermediate control horizon length where closed-loop performance is optimal, for a class of disturbances. A detailed methodology is presented in the main text. It should be emphasized that the above phenomenon of deteriorating performance for too short or too long control horizons occurs only when the *slightest* modeling uncertainty is present, which, of course, is the case for *all* practical problems. For process models that are *infinitely* accurate, the MPC methodology we propose suggests that closed-loop performance does not deteriorate as the control horizon length increases. This is reassuringly in agreement with well-known facts from the linear-quadratic regulator (LQR) theory.

The rest of this article is structured as follows: We first formulate the multivariable EQDMC algorithm, and present robust stability and performance results for multivariable EQDMC. We then summarize our results in a tuning methodology for EQDMC. Finally, we illustrate our robust MPC design method through a number of simulations.

Formulation of Multivariable EQDMC

We consider a process (assumed to be open-loop stable) modeled by a unit-pulse or unit-step response model. In EQDMC the end condition (Eq. 8 below) constrains the projected input vector in the control horizon to take a final value that drives the projected steady-state value of the process output vector to the setpoint. This is helpful in developing robust stability conditions that guarantee zero offset. Also present in the EQDMC formulation are input constraints and soft output constraints, so that feasibility of the on-line optimization problem can be guaranteed. The full on-line optimization problem to be solved at time k is cast as follows:

$$\min_{\epsilon(k+1),\ldots,\epsilon(k+nw),\Delta \boldsymbol{u}(k),\Delta \boldsymbol{u}(k+1),\ldots,\Delta \boldsymbol{u}(k+p)} J(k) \quad (1)$$

with

$$J(k)=\sum_{i=1}^{no} v_i \sum_{j=1}^{nh} (\bar{y}_i(k+j)-y_i^{sp})^2 + \sum_{i=1}^{no} W_i \sum_{j=1}^{nw} \epsilon_i^2(k+j) + \sum_{i=1}^{ni} \sum_{j=0}^{p} r_{i,j} \Delta u_i^2(k+j) \quad (2)$$

subject to

Process output prediction:

$$\bar{\boldsymbol{y}}(k+i)=\sum_{j=1}^{N} \boldsymbol{g}_j \boldsymbol{u}(k+i-j)+\bar{\boldsymbol{d}}(k+i); \quad (3)$$

Disturbance prediction:

$$\bar{\boldsymbol{d}}(k+i)=\bar{\boldsymbol{d}}(k)=\boldsymbol{y}(k)-\sum_{j=1}^{N} \boldsymbol{g}_j \boldsymbol{u}(k-j)=\boldsymbol{d}(k) + \sum_{j=1}^{N} \boldsymbol{e}_j \boldsymbol{u}(k-j); \quad (4)$$

Input move constraint:

$$\Delta u_{l_{\max}} \ge \Delta u_l(k+i) \ge -\Delta u_{l_{\max}}, \quad 1 \le l \le ni; \quad (5)$$

Input constraint:

$$u_{l_{\max}} \ge u_l(k+i) \ge u_{l_{\min}}, \quad 1 \le l \le ni, \quad \text{for} \quad i=0,1,\ldots,p; \quad (6)$$

Softened output constraint:

$$y_{i_{\max}}+\epsilon_i(k+j) \ge \bar{y}_i(k+j) \ge y_{i_{\min}}-\epsilon_i(k+j), \quad 1 \le i \le no, \quad j=1,2,\ldots,nw; \quad (7)$$

End condition:

$$\boldsymbol{u}(k+p+i)=\boldsymbol{G}^{-1}(\boldsymbol{y}^{sp}-\bar{\boldsymbol{d}}(k)), \quad i \ge 0 \quad (8)$$

where

no = number of outputs; $ni(=no)$, number of inputs
nh = prediction horizon length
p = control horizon length
nw = soft constraint horizon length
N = number of past inputs used in the pulse response model
$J(k)$ = on-line objective function to be minimized at time k
$\boldsymbol{G}$ = steady-state $no \times ni$ gain matrix of the multivariable plant

h_i, g_i = $no \times ni$ matrices containing the ith unit-pulse response coefficients for the real plant and the plant model, respectively
$e_j = h_j - g_j$ = modeling error
$y(k)$ = actual output column vector of dimension no at time k
$y_{i\max}$, $y_{i\min}$ = upper and lower bounds on $y_i(k)$, respectively
$\bar{y}(k+i)$ = predicted output vector at time $k+i$
y^{sp} = output setpoint vector of dimension no
$u(k)$ = input column vector of dimension ni at time k
$u_{l\max}$, $u_{l\min}$ = upper and lower bounds on $u_l(k)$, respectively
$\Delta u(k+i) = u(k+i) - u(k+i-1)$ = control move vector
$\Delta u_{l\max}$, $\Delta u_{l\min}$ = upper and lower bounds on $\Delta u_l(k)$, respectively
$\bar{d}(k)$ = predicted disturbance vector of dimension no, at time k
$d(k)$ = actual disturbance at time k
$\epsilon_i(k+j)$ = constraint violation variable for the ith output prediction at time $k+j$: $\epsilon_i(k+j) = \max\{0, [y_i(k+j) - y_{i,\max}], [y_{i,\min} - y_i(k+j)]\}$
v_i = weights for the output deviation term in the objective function
w_i = weights for the output constraint violation terms in the objective function
$r_{i,j}$ = move-suppression weight terms for the jth move and the ith input in the objective function

We take the real output $y(k)$ to be

$$y(k) = d(k) + \sum_{j=1}^{N} h_j u(k-j) \tag{9}$$

Robust Stability of Multivariable EQDMC

Theorem (Genceli, 1993)

For a plant described by Eq. 9, and external disturbances $d(k)$ such that for a certain $M > 0$,

$$d_{\min} \le d(k) \le d_{\max} \tag{10}$$

$$|\Delta d(k)| \le \Delta d_{\max} \tag{11}$$

where $\Delta d_{\max} > 0$ if $k \le M$; $\Delta d_{\max} = 0$, that is, $d(k) = d_\infty$, if $k > M$, the closed-loop multivariable EQDMC system described by the set of Eqs. 1 to 8, is stable with zero offset if the following conditions are satisfied:

(i) G has full rank (no);

(ii) The control horizon, prediction horizon, and constraint window lengths ($p+1$, nh, and nw) satisfy the inequalities

$$nc \ge p+1 \ge p_{\min} = \max\left\{\frac{u_{i_{\max}} - u_{i_{\min}}}{\Delta u_{i_{\max}}}, \quad 1\right\} - 1, \quad i = 1, \ldots, ni \tag{12}$$

where $nc = \min\{nh-1, nw-1\}$;

(iii) Process modeling and disturbance uncertainty satisfy the conditions

$$\text{(iii.a)} \sum_{l=1}^{no} \max\{G_{il}u_{l_{\max}}, G_{il}u_{l_{\min}}\} \ge y_i^{sp} - d_{i_{\min}} + \sum_{l=1}^{no}\left[U_l \sum_{j=1}^{N} E_{jil}\right]; \tag{13a}$$

$$\text{(iii.b)} \sum_{l=1}^{no} \min\{G_{il}u_{l_{\max}}, G_{il}u_{l_{\min}}\} \ge y_i^{sp} - d_{i_{\max}} + \sum_{l=1}^{no}\left[U_l \sum_{j=1}^{N} E_{jil}\right]; \tag{13b}$$

$$\text{(iv)} \frac{\min\limits_{l}(\Delta u_{l_{\max}})}{\|G^{-1}\|_\infty} - \max_l\{\Delta u_{l_{\max}}\} \sum_{j=1}^{N} \|E_j\| \ge \max_l(\Delta d_{i_{\max}}) > 0; \tag{14}$$

where $1 \le i \le no$, $1 \le l \le no$, $1 \le j \le N$; $U_l = \max\{|u_{l_{\max}}|, |u_{l_{\min}}|\}$; $|e_{jil}| \le E_{jil}$, and E_j is a square matrix with entries E_{jil} that place an upper bound on the modeling error e_{jil};

(v) The move suppression terms $\{r_{ij}\}_{i=1,\ldots,ni \,\&\, j=0,\ldots,p}$ are calculated from the recursive formulas;

$$r_p L = [I - W^T D]^{-1}\left(\sum_{j=-N+1}^{p}\left[C_j^* + \hat{C}_j^*\right] + \delta L\right) \tag{15}$$

and

$$r_j L = \left[r_{j-1} + |G^{-1}E_{1-j}|^T r_p \sum_{s=-N+1}^{0} |G^{-1}E_{1-s}|\right] L^T + C_j^* + \hat{C}_j^* + \delta L, \quad 0 \le j \le p, \tag{16}$$

where

δ = small positive number

$$r_j = \text{diag}\{r_{1,j} \ldots r_{ni,j}\};$$

L = column vector of dimension ni whose entries are identically 1;

$$W = \sum_{i=1}^{N} |G^{-1}E_i|; \tag{17}$$

D = diagonal matrix with

$$D_{ii} = \sum_{j=1}^{no} W_{ij}; \tag{18}$$

C_j^*, $\hat{C}_j^*$ = no dimensional vectors whose nth terms are

$$[C_j^*]_n = \sum_{r=1}^{no} \sum_{s=-N+1}^{p} |C_{n,r,j,s}|, \qquad [\hat{C}_j^*]_n = \sum_{r=1}^{no} \sum_{s=-N+1}^{p} |\hat{C}_{n,r,j,s}| \tag{19}$$

where

$C_{n,r,j,s}$, $\hat{C}_{n,r,j,s}$ = (n,j)th term of the matrices $C_{r,s}$ and $\hat{C}_{r,s}$ given by

$$\begin{aligned} C_{r,s} &= 0.5(X_{r,s} + X_{r,s}^T) \\ \hat{C}_{r,s} &= 0.5\left(\hat{X}_{r,s} + \hat{X}_{r,s}^T\right) \end{aligned} \tag{20}$$

with

$$\begin{aligned} X_{r,s} = &\sum_{i=0}^{p} E_{1-r}^T v^T v(E_r - 2Q(i,s)) \\ &+ \sum_{i=p+1}^{nh-1} E_{1-r}^T F^T(i) v^T v F(i) E_{1-s} - 2Q^T(i,s) v^T v F(i) E_{1-r} \\ &+ (F(nh)E_{1-r} - Q(nh,r))^T v^T v (F(nh)E_{1-s} - Q(nh,s)), \\ \hat{X}_{r,s} = &\sum_{i=0}^{nw} E_{1-r}^T F^T(i) w^T w F(i) E_{1-s} \\ &+ 2[|Q(i,s)| - \gamma(i,j)]^T w^T w F(i) E_{1-s} \end{aligned} \tag{21}$$

with

$$F(i) = \sum_{l=1+i-p}^{N+1} g_l G^{-1} \tag{22}$$

$$Q(i,j) = \sum_{l=2+i-j}^{N+1} g_l \tag{23}$$

and $\gamma_{m,n}(i,j)$'s chosen such that

$$\sum_{j=-N+1}^{p} \gamma_{m,n}(i,j) = \min\left\{ y_m, |F_{m,n}(i)| \sum_{j=-N+1}^{0} E_{m,n,1-j} + \sum_{j=-N+1}^{p} |Q_{m,n}(i,j)| \right\} \tag{24}$$

and

$$y_m = \max\{0, (y_m^{sp} - y_{m,\max}), (y_{m,\min} - y_m^{sp})\}. \tag{25}$$

Summary of proof. Let us define an augmented objective function

$$\begin{aligned} I(k) = &\sum_{i=1}^{no} \Big\{ v_i (y_i(k) - y_i^{sp})^2 \\ &+ w_i \max(0, [y_i(k) - y_{i_{\max}}], [y_{i_{\min}} - y_i(k)])^2 \Big\} \\ &+ \sum_{i=1}^{no} v_i \sum_{j=1}^{nh} (\bar{y}_i(k+j) - y_i^{sp})^2 + \sum_{i=1}^{no} w_i \sum_{j=1}^{nw} \epsilon_i^2(k+j) \\ &+ \sum_{i=1}^{ni} \sum_{j=-N+1}^{p} r_{i,j} \Delta u_i^2(k+j), \end{aligned} \tag{26}$$

which is minimized with respect to $\{\epsilon(k+1), \ldots, \epsilon(k+nw), \Delta u(k), \Delta u(k+1), \ldots, \Delta u(k+p)\}$. The on-line cost $J(k)$ is also minimized with respect to the same variables. Let the solution of the EQDMC of the augmented objective function at time k be

$$\Phi(k) = \min_{\epsilon(k+1), \ldots, \epsilon(k+nw), \Delta u(k), \Delta u(k+1), \ldots, \Delta u(k+p)} I(k) \tag{27}$$

subject to the constraints, Eqs. 3 to 8.

Conditions i and ii of the theorem guarantee a feasible solution to the preceding optimization problem.

The idea of obtaining condition v of the robust stability theorem lies in developing conditions that ensure that the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ converges to 0. To obtain these conditions, consider the optimization problem at time $k+1$, that is,

$$\Phi(k+1) = \min_{\epsilon(k+2), \ldots, \epsilon(k+nw+1), \Delta u(k+1), \Delta u(k+2), \ldots, \Delta u(k+p+1)} I(k+1) \tag{28}$$

and subject to constraints 3 to 8 at time $k+1$.

If $\Phi^*(k+1)$ is any suboptimal cost at time $k+1$, then obviously,

$$\begin{aligned} &0 \le \Phi(k+1) \le \Phi^*(k+1) \\ \Rightarrow \quad &\Phi(k) \ge \Phi(k+1) + [\Phi(k) - \Phi^*(k+1)]. \end{aligned} \tag{29}$$

Therefore, if we find a $\Phi^*(k+1)$ such that $\Phi(k) - \Phi^*(k+1) \ge 0$, then we will have $\Phi(k+1) \le \Phi(k)$. To find a suitable $\Phi^*(k+1)$, let us denote by

$$\{\hat{\epsilon}(k+1), \hat{\epsilon}(k+2), \ldots, \hat{\epsilon}(k+nw), \Delta\hat{u}(k), \Delta\hat{u}(k+1), \ldots, \Delta\hat{u}(k+p)\},$$

the solution of the optimization problem at time k. We then construct the feasible (but not necessarily optimal) solution at time $k+1$, $\{\hat{\epsilon}(k+2),\ \hat{\epsilon}(k+3), \ldots,\ \tilde{\epsilon}(k+nw+1),\ \Delta\hat{u}(k+1),\ \Delta\hat{u}(k+2), \ldots,\ \Delta\hat{u}(k+p),\ \Delta\tilde{u}(k+p+1)$, where $\Delta\tilde{u}(k+p+1)$ and $\tilde{\epsilon}(k+nw+1)$ are to be selected, so as to satisfy constraints 5 and 8. This would yield a suboptimal cost $\Phi^*(k+1)$. At this point, conditions iii and iv guarantee the existence of a suitable $\Delta\tilde{u}(k+p+1)$. After lengthy calculations, it can be shown that

$$\begin{aligned} \Phi(k) - \Phi^*(k+1) = &\sum_{i=1}^{no} \Big\{ v_i (y_i(k) - y_i^{sp})^2 \\ &+ w_i \max(0, [y_i(k) - y_{i_{\max}}], [y_{i_{\min}} - y_i(k)])^2 \Big\} \\ &+ \sum_{i=1}^{no} \sum_{j=-N+1}^{p} \delta \Delta u_i(k+j)^2 \end{aligned} \tag{30}$$

where, δ is a function of

- the move suppression terms $\{r_{-N+1}, r_{-N+2}, \ldots, r_p\}$;
- the constraint bounds $u_{\min}$, $u_{\max}$, $y_{\min}$, $y_{\max}$, $\Delta u_{\max}$, $d_{\max}$, $\Delta d_{\max}$;
- the modeling error bounds, E_i.
- the moving horizon lengths nh, nw, p.

Condition υ, Eqs. 17 to 25, of the theorem guarantees that $\delta \geq 0$, which implies

$$\Phi(k) - \Phi^*(k+1) \geq 0. \quad (31)$$

Combining the preceding inequalities with Eq. 29, we get

$$\Phi(k) \geq \Phi(k+1) \geq 0, \quad (32)$$

that is, the sequence $\{\Phi(k)\}_{k=0}^{\infty}$ is convergent. From Eqs. 29 and 30 we get

$$\Phi(k) \geq \Phi(k+1) + \sum_{i=1}^{no} \Big\{ \upsilon_i (y_i(k) - y_i^{sp})^2 + w_i \max(0, [y_i(k) - y_{i_{\max}}], [y_{i_{\min}} - y_i(k)])^2 \Big\} + \sum_{i=1}^{no} \sum_{j=-N+1}^{p} \delta \Delta \boldsymbol{u}_i(k+j)^2. \quad (33)$$

Taking limits on both sides yields

$$0 \geq \sum_{i=1}^{no} \Big\{ \upsilon_i (y_i(k) - y_i^{sp})^2 + w_i \max(0, [y_i(k) - y_{i_{\max}}], [y_{i_{\min}} - y_i(k)])^2 \Big\} + \sum_{i=1}^{no} \sum_{j=-N+1}^{p} \delta \Delta \boldsymbol{u}_i(k+j)^2 \geq 0, \quad (34)$$

which implies that $y(\infty) = y^{sp}$, that is, zero-offset control.

The complete proof is available in Supplementary Information.

This theorem provides a sufficient condition for the robust stability of the closed-loop EQDMC system, and provides a methodology for calculating the size of move-suppression terms. A choice of values for the move-suppression terms equal to or higher than the values dictated by the theorem would guarantee closed-loop stability. It is, however, important to note that a choice smaller than these bounds on the move-suppression terms does not imply instability, as this theorem provides only a sufficient condition for stability.

Remark. In practice, there is a possibility that disturbances violating Eqs. 10 and 11 may enter a process. In that case, the EQDMC optimization problem will not have a feasible solution because of the equality constraint, Eq. 8. Under such circumstances, we may select

$$\begin{aligned} u_i(k+p+j) &= u_{i_\infty}, \quad \text{if } u_{i_{\min}} \leq u_{i_\infty} \leq u_{i_{\max}}, \quad j \geq 0 \\ u_i(k+p+j) &= u_{i_{\min}}, \quad \text{if } u_{i_\infty} \leq u_{i_{\min}}, \quad j \geq 0 \\ u_i(k+p+j) &= u_{i_{\max}}, \quad \text{if } u_{i_\infty} > u_{i_{\max}}, \quad j \geq 0 \end{aligned} \quad (35)$$

where $u_{i\infty}$ is ith input, given by the ith element of the vector $\boldsymbol{u}_\infty$ given by

$$\boldsymbol{u}_\infty = \boldsymbol{G}^{-1}(\boldsymbol{y}^{sp} - \bar{\boldsymbol{d}}(k)). \quad (36)$$

Robust Performance of Multivariable EQDMC

We first define a performance index that can be used in the selection of optimal values for the controller parameters.

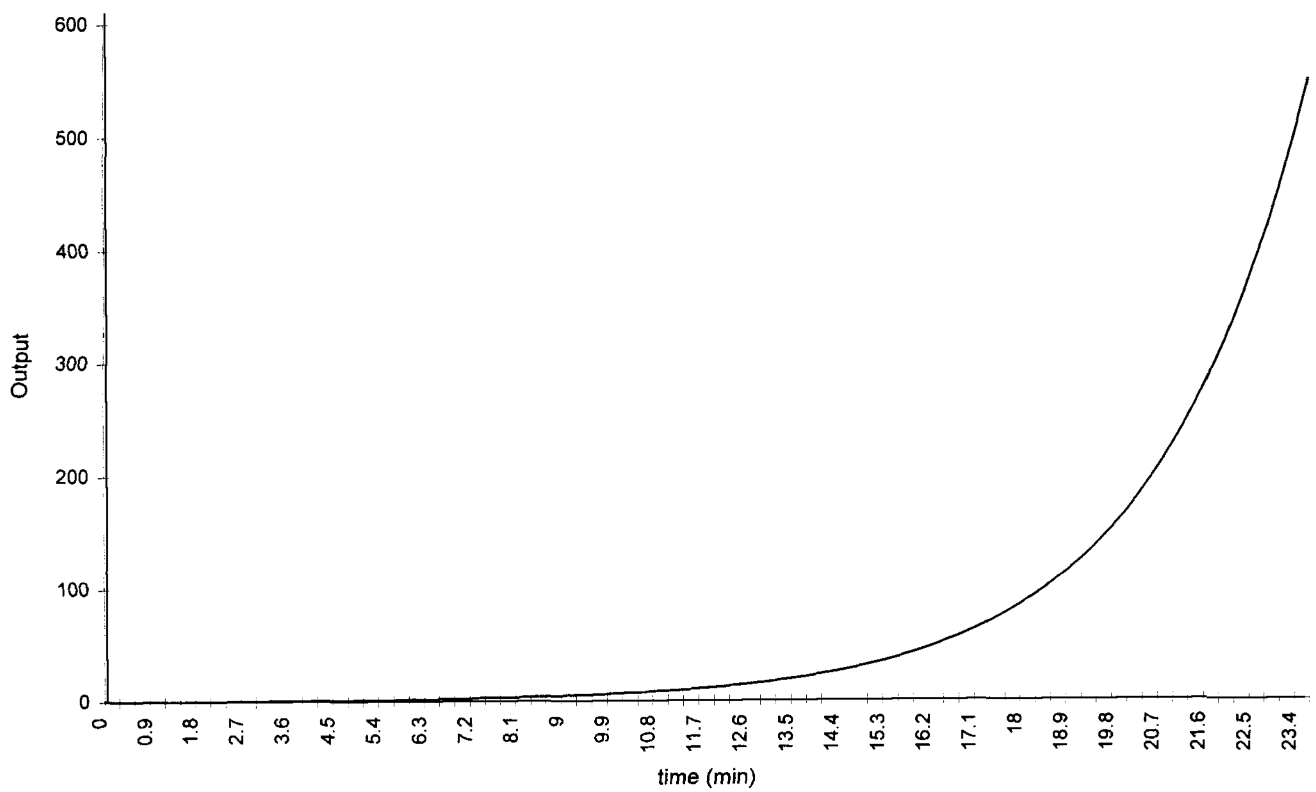

Figure 1. Closed-loop response to a step disturbance for Example 1, Design A.

Then, based on that index, we show how to select the best performing controller out of the class of controllers that guarantee robust closed-loop stability.

Definition. We define the performance index, P, for a multivariable EQDMC closed-loop system by

$$P = \sum_{i=1}^{no} \sum_{k=0}^{\infty} \Big\{ v_i (y_i(k) - y_i^{sp})^2 + w_i \max(0, [y_i(k) - y_{i_{\max}}], [y_{i_{\min}} - y_i(k)])^2 \Big\}, \quad (37)$$

where y_i is the ith measured process output, v_i and w_i are the weights for the deviation of the ith output from the set-point and the output constraint violation terms, respectively, in the cost function $J(k)$ (Eq. 2).

Corollary (robust performance of EQDMC, Genceli (1993))

The EQDMC controller, defined by the set of relations 1 to 8, achieves a closed-loop performance, P, bounded as follows:

(a) For step-disturbances

$$P \leq \min_{\epsilon_1, \ldots, \epsilon_{nw}, \Delta u(0), \Delta u(1), \ldots, \Delta u(p)} J(0) \quad (38)$$

where

$$J(0) = \sum_{i=1}^{no} v_i \sum_{j=1}^{nh} (\bar{y}_i(j) - y_i^{sp})^2 + \sum_{i=1}^{no} w_i \sum_{j=1}^{nw} \epsilon_i^2(k+j) + \sum_{i=1}^{ni} \sum_{j=0}^{p} r_{i,j} \Delta u_i^2(j) \quad (39)$$

(b) For arbitrary disturbances that satisfy the conditions of the theorem,

$$P \leq \left[\max_{\epsilon_1, \ldots, \epsilon_{nw}, \Delta u(0), \ldots, \Delta u(p)} J(0) \right] + f(0) \quad (40)$$

where

$$f(0) = (\beta + \hat{\beta}) \Delta d_{\max}^2 [M+1] \quad (41)$$

$$\beta = \frac{1}{4\delta} \left(\sum_{n1=1}^{no} \sum_{n1=1}^{no} \left[|A + G^{-1^T} r_p G^{-1}| + \sum_{j=-N=1}^{p} \left(|B_j + G^{-1^T} r_p G^{-1} E_{1-j}| \right)_{n1,n2} \right] \right)^2 \quad (42)$$

and

$$\hat{\beta} = \frac{1}{4\delta} \left(\sum_{n1=1}^{no} \sum_{n2=1}^{no} \left[|\hat{A} + G^{-1^T} r_p G^{-1}| + \sum_{j=-N+1}^{p} \left(|\hat{B}_j + G^{-1^T} r_p G^{-1} E_{1-j}| \right)_{n1,n2} \right] \right)^2 \quad (43)$$

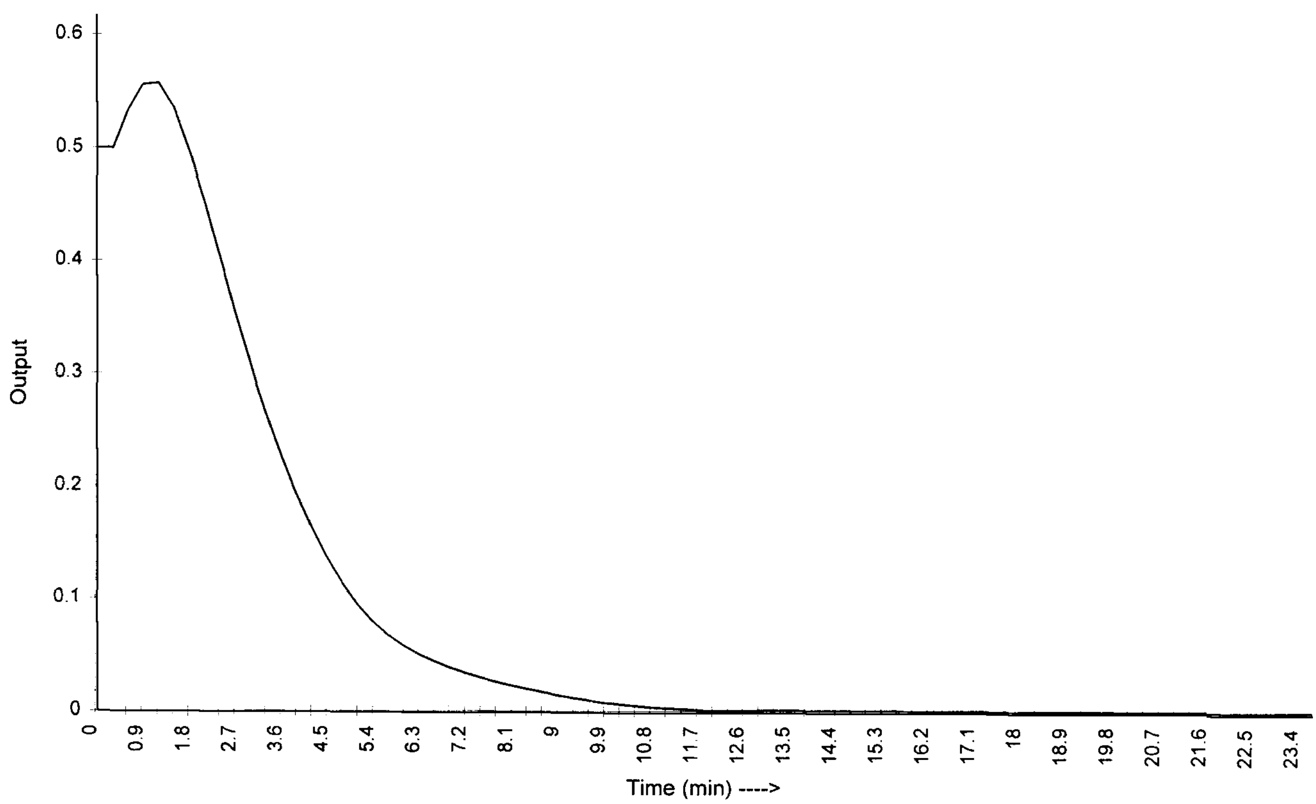

Figure 2. Closed-loop response to a step disturbance for Example 1, Design B.

with

$$A = (p+1)v^T v + F^T(nh)v^T vF(nh) - \sum_{i=p+1}^{nh-1} F^T(i)v^T vF(i) \quad (44)$$

$$\hat{A} = (p+1)w^T w + \sum_{i=p+1}^{nh-1} F^T(i)w^T wF(i) \quad (45)$$

$$B_j = 0.5\left(Y_j^T + Y_j\right) \quad \text{and} \quad \hat{B}_j = 0.5\left(\hat{Y}_j^T + \hat{Y}_j\right) \quad (46)$$

$$Y_j = \sum_{i=1}^{nh-1} 2\left(F^T(i)v^T vQ(i,j) - F^T(i)v^T vF(i)E_{1-j}\right) + 2F^T(nh)v^T v(F(nh)E_{1-j} - Q(nh,j)) \quad -N+1 \le j \le p \quad (47)$$

and

$$\hat{Y}_j = \sum_{i=1}^{nw} 2\left(F^T(i)w^T w[|Q(i,j)| - \gamma(i,j)] - F^T(i)w^T wF(i)E_{1-j}\right). \quad (48)$$

Proof. Given below is the proof for the EQDMC controller for step disturbances. Consider Eq. 30. Choosing $\delta = 0$ results in

$$\Phi(k) \ge \Phi(k+1) + \sum_{i=1}^{no} \left\{ v_i(y_i(k) - y_i^{sp})^2 + w_i \max(0,[y_i(k) - y_{i_{\max}}],[y_{i_{\min}} - y_i(k)])^2 \right\}. \quad (49)$$

Successive substitutions of $\Phi(k)$ in the preceding equation, starting from the initial time $k = 0$, yields

$$\Phi(0) \ge \sum_{k=0}^{\infty} \sum_{i=1}^{no} \left\{ v_i(y_i(k) - y_i^{sp})^2 + w_i \max(0,[y_i(k) - y_{i_{\max}}],[y_{i_{\min}} - y_i(k)])^2 \right\} = P \quad (50)$$

The proof for the cases with more general disturbances, rely on the same idea, and can be found in Supplementary Information.

In the case of general disturbances, too small a value for δ would mean large values for β and $\hat{\beta}$, as can be seen from Eqs. 42 and 43. This would give a conservative bound for P. A larger value for δ would increase move suppression, making the controller more sluggish, thereby deteriorating performance and resulting in a larger value for P. However, the bound on P would be tighter.

As illustrated in the next article, in the absence of modeling errors and in the presence of constant disturbances, the performance of the closed-loop EQDMC system improves as the control horizon p increases, prompting the choice of as large a value for p as possible for $nw = nh = p + N - 1$. This is because in the complete absence of modeling errors all the

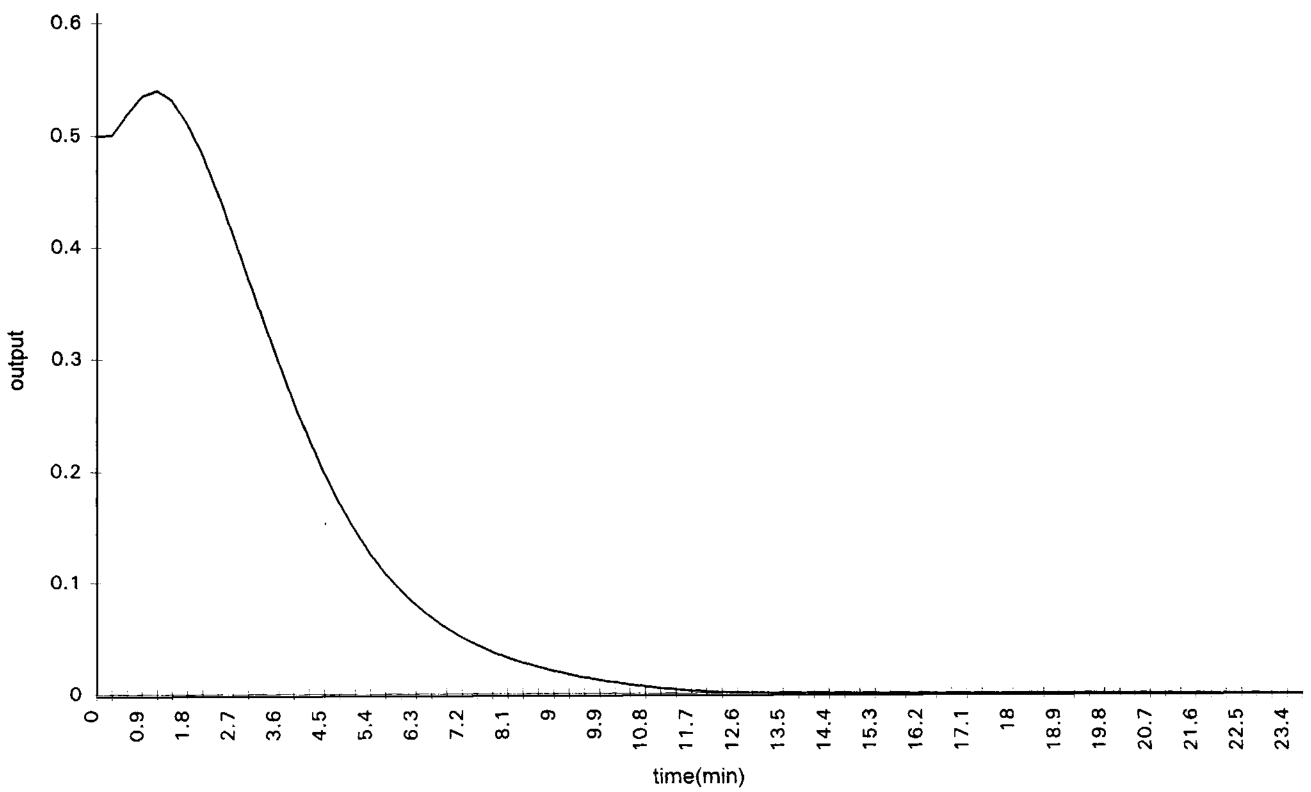

Figure 3. Closed-loop response to a step disturbance for Example 1, Design C.

move-suppression terms become zero, and hence a larger p would mean a quicker closed-loop response. However, process modeling errors are always present. In such cases, starting from $p = p_{\min}$ and increasing p may initially improve performance. But increasing p beyond a particular value often means larger values for the move-suppression terms r_p, $r_{p-1}, \ldots, r_0$, as we have observed in a number of simulations. One could justify this observed behavior as follows: For very small p, aggressive control action should be penalized to prevent instability. Therefore r_i should be large. As p increases r_i could initially decrease. However, as p becomes large, uncertainty becomes an important factor in the following sense: To guarantee stability for all possible future outputs (corresponding to all possible process models), the controller must be very cautious, that is, it must allow very little control action. This is because these move-suppression terms are rather sensitive to increases in p and modeling error. Therefore, choosing a larger p results in a more conservative controller (since the move-suppression terms have a damping effect on the response of the process). Hence, in the presence of modeling error, performance is expected to reach an optimum value for an intermediate value of p. Values of p above or below its optimum would result in performance deterioration. The sensitivity of the optimal performance to changes close to the optimum p is rather small, as shown in subsequent simulations.

EQDMC Tuning Methodology

An EQDMC tuning methodology that results in optimal performance in the presence of uncertainty is summarized below:

Step 0. Select values for $y_{i_{\min}}$, $y_{i_{\max}}$, $u_{i_{\min}}$, $u_{i_{\max}}$, $\Delta u_{i_{\max}}$, $d_{i_{\max}}$, $\Delta d_{i_{\max}}$, v, w.

Step 1. Choose $p \geq p_{\min}$, where $p_{\min}$ is calculated from Eq. 12.

Step 2. Choose nh and nw so as to satisfy Eq. 12.

Step 3. Calculate the move suppression terms as follows:

- Compute vectors $\boldsymbol{C}_j^*$ and $\hat{\boldsymbol{C}}_j^*$ using Eqs. 19 to 25.
- Determine matrices $\boldsymbol{W}$ and $\boldsymbol{D}$ using Eqs. 17 and 18.
- Obtain move suppression terms for the $(p+1)$-th move of the ni inputs, that is, the terms of $\boldsymbol{r}_p$ from Eq. 15.
- Knowing $\boldsymbol{r}_p$ get $\boldsymbol{r}_{p-1}$ from Eq. 16. Continue determining $\boldsymbol{r}_{j-1}$ from $\boldsymbol{r}_j$ up to $\boldsymbol{r}_0$.

Step 4. If enough values of p have been examined, to reveal an optimum, stop. Else, go to 1.

Remarks:

- Although step 0 is straightforward, it is crucial for the ensuing controller design. In fact, knowledge of $d_{i_{\max}}$ and $\Delta d_{i_{\max}}$ may not be readily available, and good engineering judgment may be necessary.
- The optimization with respect to p, performed by the preceding algorithm, is computationally inexpensive. Indeed a single integer variable, p, is involved. Therefore an exhaustive search can easily yield the optimum. From experience we have found that the J vs. p curve shows an increasing trend, after an apparent global optimum, that we have found to be around the value

$$p = \max\left(p_{\max}, \{N_{d_{ij}}\}_{i=1,j=1}^{no,ni}\right) \tag{40}$$

where $N_{d_{ij}}$ is either the dead time or, for systems with in-

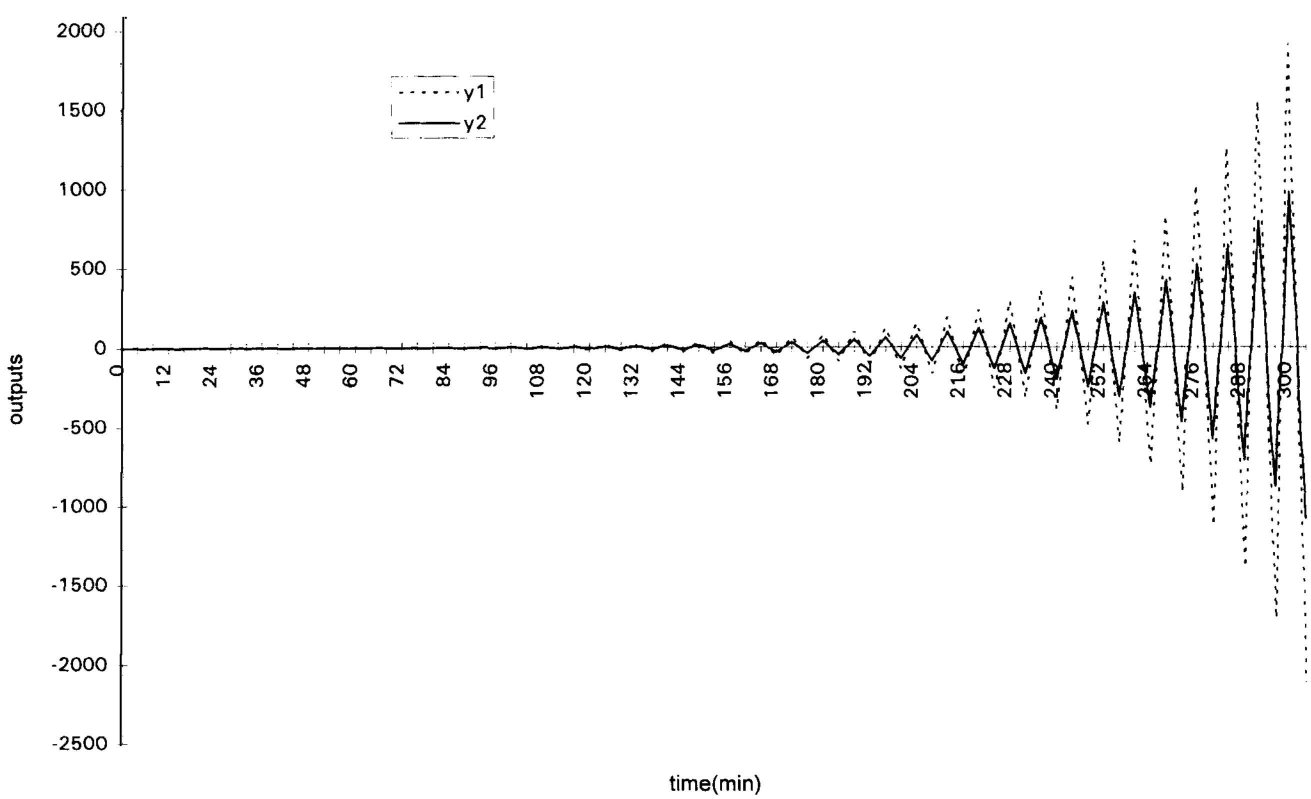

Figure 4. Closed-loop response to step disturbances for Example 2, Design A.

verse response, the time when the pulse response curve first changes sign.

- It can be intuitively justified that the optimal values for nh and nw in step 2 are $nh = nw = N + p$. This was verified in our simulations, where we examined the sensitivity of r_0 to changes in nh.

Simulations

In the following simulations we wish to demonstrate:

- The stabilizing effect of the end condition on an otherwise unstable closed-loop with conventional constrained DMC.
- The effect of the control horizon p on the performance of closed-loop EQDMC systems that satisfy the robust stability conditions of the preceding theorem, for different levels of modeling error.

The simulations are performed for a SISO problem and a 2×2 subsystem of the Shell Standard Control Problem (Prett and Garcia, 1988). For both examples, we first demonstrate the stabilizing effect of the end condition by making comparisons between the stability and performance of QDMC and EQDMC systems, that is, with and without the end condition introduced, everything else remaining the same. Then, EQDMC controllers are tuned with the move-suppression terms calculated so as to satisfy Genceli and Nikolaou's (1993b) conditions and their multivariable generalizations presented herein. Finally, different prediction and control horizon lengths of robustly stabilizing EQDMC controllers are considered, in order to obtain the best performance while ensuring closed-loop stability.

Example 1 (SISO)

The plant model transfer function is

$$\bar{P}(s) = \frac{(-s+1)e^{-0.3s}}{(s+1)(2s+1)}$$

where the real plant is simulated by

$$y = Pu + d.$$

We use a sampling interval of 0.3 and a settling time of $N = 40$ to get the pulse response model. We consider an output constraint of $-0.3 \leq y \leq 0.3$, and use $p = 2$, $nh = nw = 7$. The plant has a 10% maximum modeling error in each coefficient h_i of the unit pulse response model, that is, $E_i = 0.1 h_i$. We consider different controller designs. For design A we use a QDMC controller with *arbitrary* move-suppression values $r_2 = 1.5$, $r_1 = 1.2$, $r_0 = 1.0$. In design B we add the end condition to design A with everything else remaining the same. In design C we use the stable EQDMC system, that is, design B with move-suppression terms set so as to guarantee robust stability according to the stability conditions developed in Genceli and Nikolaou (1993b). These three designs are summarized in Table 1.

The responses of outputs to a step disturbance of size 0.5 for each of these cases are given in Figures 1, 2, and 3, respectively. As can be seen from these simulations, design A is unstable. The end condition in design B does have a stabilizing effect on the response and case C is *guaranteed* to be stable at the cost of lower performance.

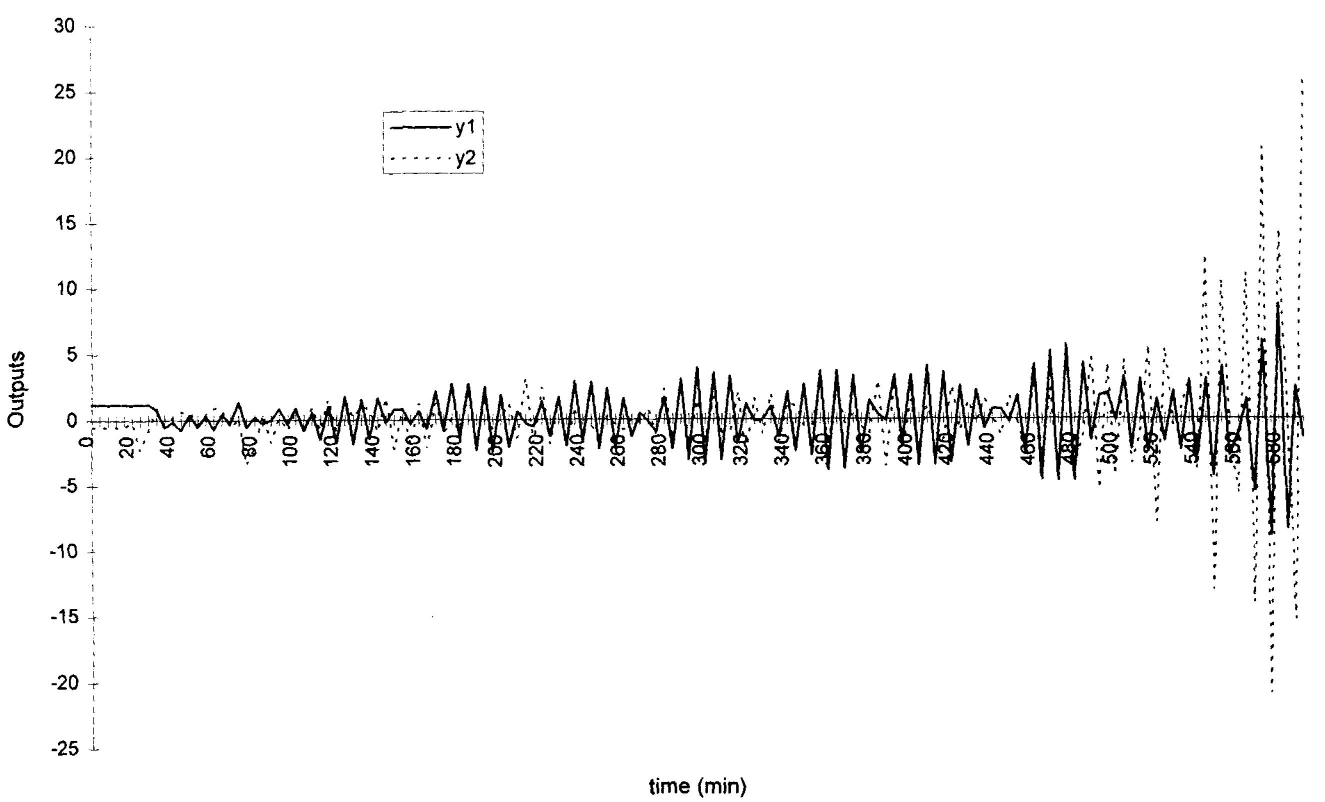

Figure 5. Closed-loop response to step disturbances for Example 2, Design B.

Table 1. Design Summary for Example 1

Design	End Condition	Move-Suppression Coefficients	Stability	P
A	Absent	$r_2 = 1.5$ $r_1 = 1.2$ $r_0 = 1.0$	Unstable	∞
B	Present	$r_2 = 1.5$ $r_1 = 1.2$ $r_0 = 1.0$	Stable, but stability *not guaranteed*	2.33
C	Present	$r_2 = 7.58$ $r_1 = 7.06$ $r_0 = 6.46$	Stability *guaranteed*	2.70

Example 2 (MIMO)

We consider the top 2×2 subsystem of the heavy oil fractionator modeled in the Shell Standard Control Problem (Prett and Garcia, 1988) as

$$\bar{P}(s) = \begin{bmatrix} \dfrac{4.05e^{-27s}}{50s+1} & \dfrac{1.77e^{-28s}}{60s+1} \\ \dfrac{5.39e^{-18s}}{50s+1} & \dfrac{5.72e^{-14s}}{60s+1} \end{bmatrix}$$

also examined in Zafiriou (1990). We use a sampling interval of 4 min and settling time of $N = 80$ to get the pulse response model. We use the following (modified) constraints:

$$-3.0 \leq \Delta u_2(k) \leq 3.0,$$
$$-5.0 \leq u_2(k) \leq 5.0,$$
$$-0.5 \leq y_1(k+7) \leq 0.5$$

with setpoints

$$y_1^{SP} = y_2^{SP} = 0.0$$

and step disturbances

$$d_1 = 1.2, \quad d_2 = -0.5.$$

We choose

$$p = 3, \quad nh = nw = 7$$

and consider four controller designs, designs A, C, and D are similar to those in Example 1. In this example we introduce another design B. These designs are summarized in Table 2.

Our simulations show a behavior similar to that in the SISO case, for each of the three designs (A, C, and D). This can be observed in Figures 4, 6, and 7. In design B we use the end condition, but the move suppression used renders the system unstable (Figure 5). A comparison of designs B and D demonstrates the increased robustness of our proposed design D. We find that in design C here, $y_1(k)$ does not go to y^{sp}, but in fact oscillates around it. Clearly the end condition is attempting to drive the output to y^{sp}, but the move suppression is not adequate.

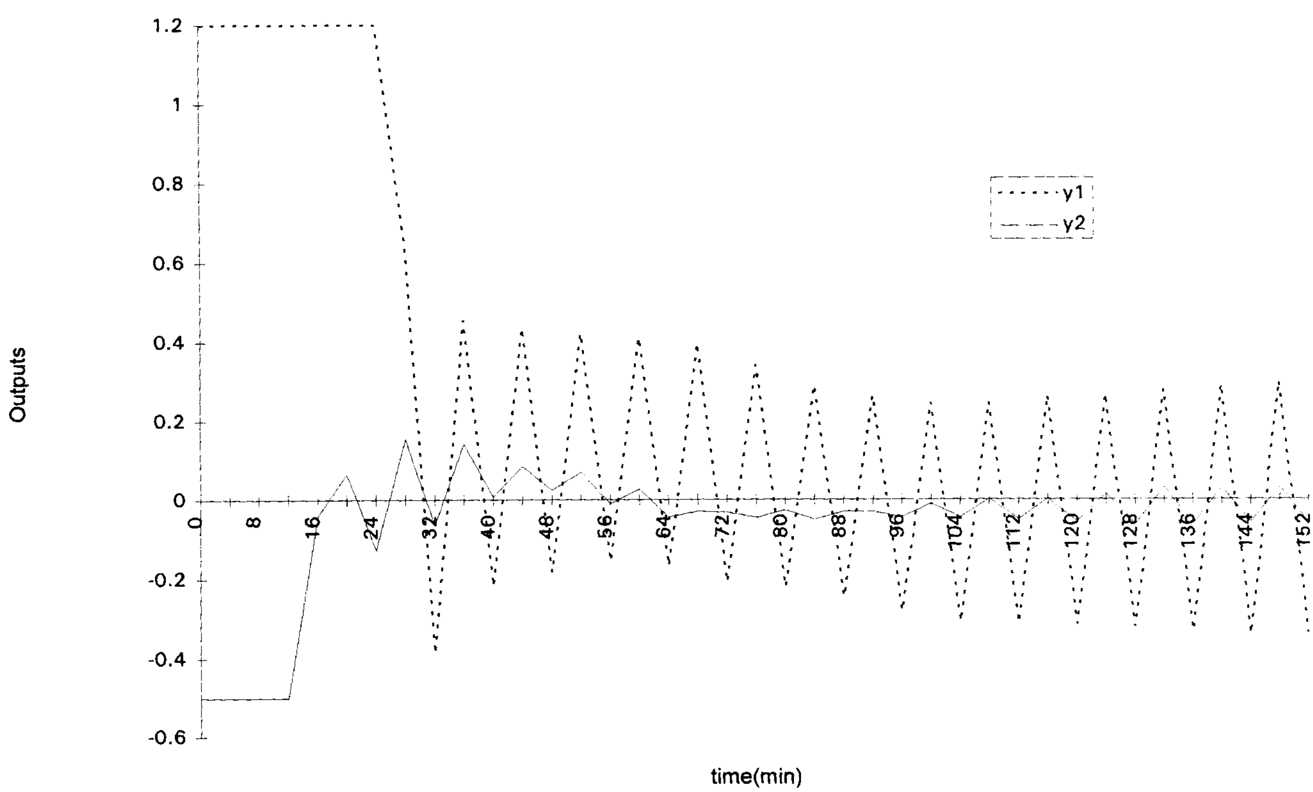

Figure 6. Closed-loop response to step disturbances for Example 2, Design C.

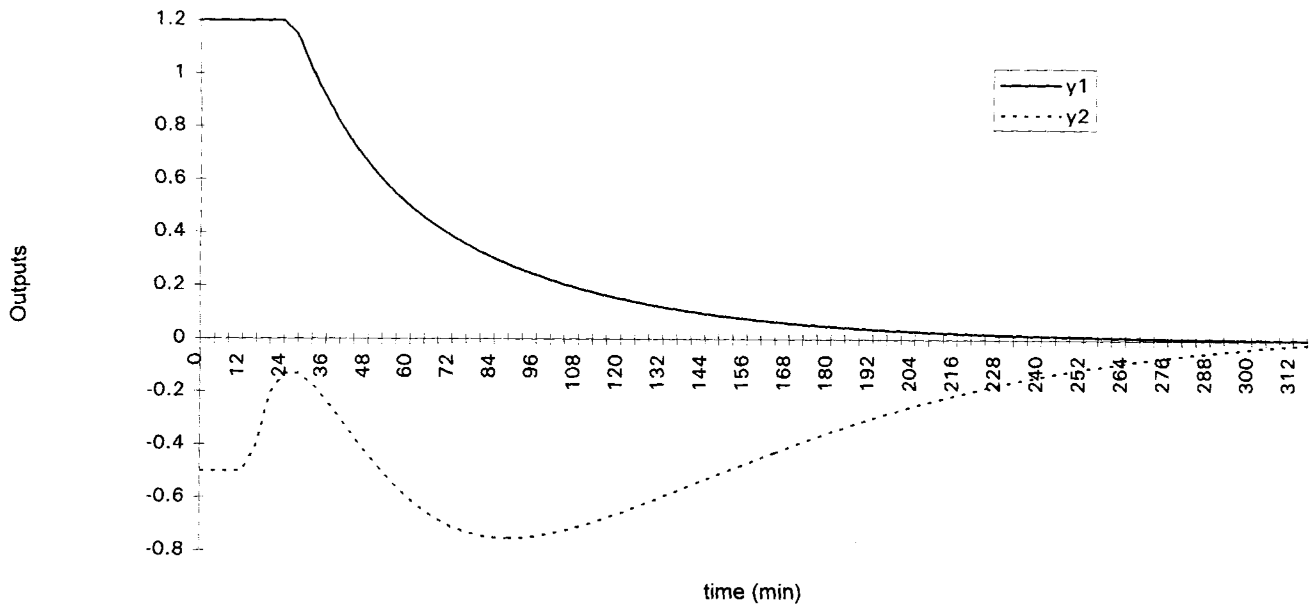

Figure 7. Closed-loop response to step disturbances for Example 2, Design D.

It is quite obvious from the preceding designs that the reason for design A in both examples being unstable is that very short prediction horizons are chosen. We have demonstrated here the effectiveness of the end condition and enhanced robustness of our design, even for short horizons. It is true that this need not be the only way to make a system stable, but is an efficient way of doing so. One way to render design A stable is by choosing a longer prediction horizon. Using the end condition would, however, greatly reduce the computational load, specifically in the presence of short output constraints.

Performance

So far the simulations have demonstrated the effect of the end condition on the performance of constrained QDMC systems. The following simulations seek to observe the dependence of performance on the prediction and control horizons. The move suppression terms play a very important role in the speed of response of the system. The move-suppression terms are calculated for the plant in Example 1 with maximum modeling error $E_i = 5\%$ of g_i for values of p from 1 to 12 and nh from $p+2$ to $p+N$. A three-dimensional plot of r_0 vs. p and nh is shown in Figure 8. From this we see that the move-suppression values are not too sensitive to the prediction horizon nh at large values of nh. As for the control horizon p, we observe that it takes a minimum value corresponding to $p = N_d$, where N_d corresponds to the N_dth term, where there is a change of sign in the process pulse response model, that is, $g_{Nd-1} \leq 0$ and $g_{Nd} \geq 0$. For minimum phase systems it is at $p = 1$. Figure 9 shows how the move-suppression terms vary with change in modeling error

Table 2. Design Summary for Example 2

Design	End Condition	Move-Suppression Coefficients	Stability	P
A	Absent	$r_{1,0} = r_{2,0} = 0.10$ $r_{1,1} = r_{2,1} = 0.12$ $r_{1,2} = r_{2,2} = 0.12$ $r_{1,3} = r_{2,3} = 0.12$	Unstable	∞
B	Present	$r_{1,0} = r_{2,0} = 0.06$ $r_{1,1} = r_{2,1} = 0.07$ $r_{1,2} = r_{2,2} = 0.07$ $r_{1,3} = r_{2,3} = 0.07$	Unstable	∞
C	Present	$r_{1,0} = r_{2,0} = 0.10$ $r_{1,1} = r_{2,1} = 0.12$ $r_{1,2} = r_{2,2} = 0.12$ $r_{1,3} = r_{2,3} = 0.12$	Stable but stability *not guaranteed*	14.85
D	Present	$r_{1,0} = 9.89$, $r_{2,0} = 10.82$ $r_{1,1} = 10.14$, $r_{2,1} = 11.15$ $r_{1,2} = 10.38$, $r_{2,2} = 11.46$ $r_{1,3} = 10.63$, $r_{2,3} = 11.82$	*Guaranteed* Stability	25.51

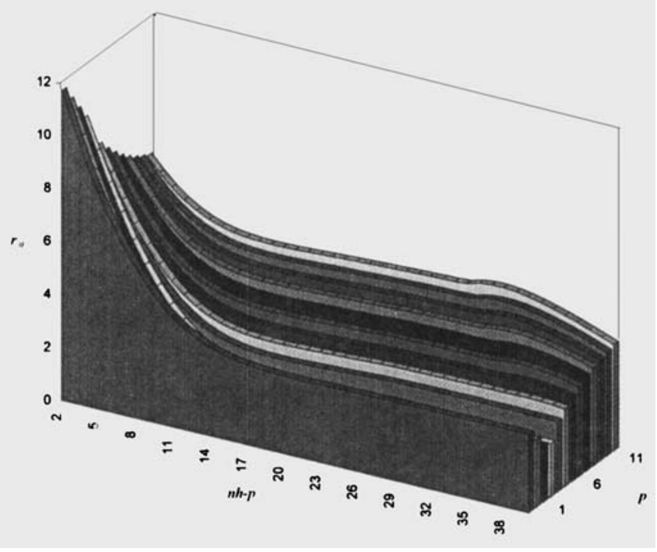

Figure 8. Dependence of the move-suppression term, r_0, on p and $nh-p$, Example 1.

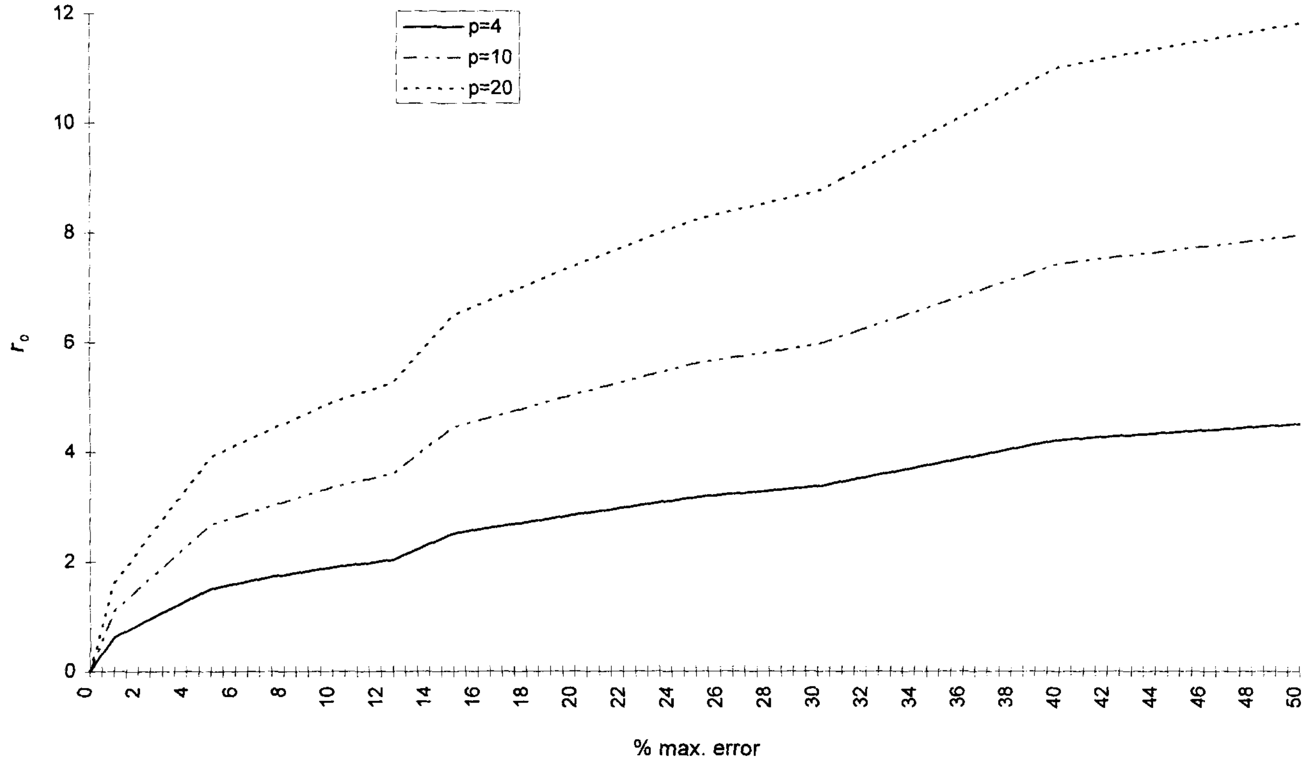

Figure 9. Dependence of the move-suppression term, r_0, on the error bound, E_i, for various p, Example 1.

E_i for different control horizon lengths p. Figure 10 demonstrates the behavior of ri vs. i for different control horizons for Example 1, and Figures 11 and 12 demonstrate the behavior of $r_{i,1}$ and $r_{i,2}$ vs. i, respectively, for different control horizons for Example 2.

We then simulated the plant for different levels of error for p between $p_{\min}$ and N with $nh = nw = p + N$. Plots of performance index P vs. control horizon length p are given in Figure 13 for Example 1 and Figure 14 for Example 2.

In the plot of P vs. p for the plant in Example 1, we see that, for the case with no modeling error, P improves with increasing p. But with modeling errors, even as low as $E_i =$ 5% of g_i, we find that performance is best for a low p, that is, $p = 4$.

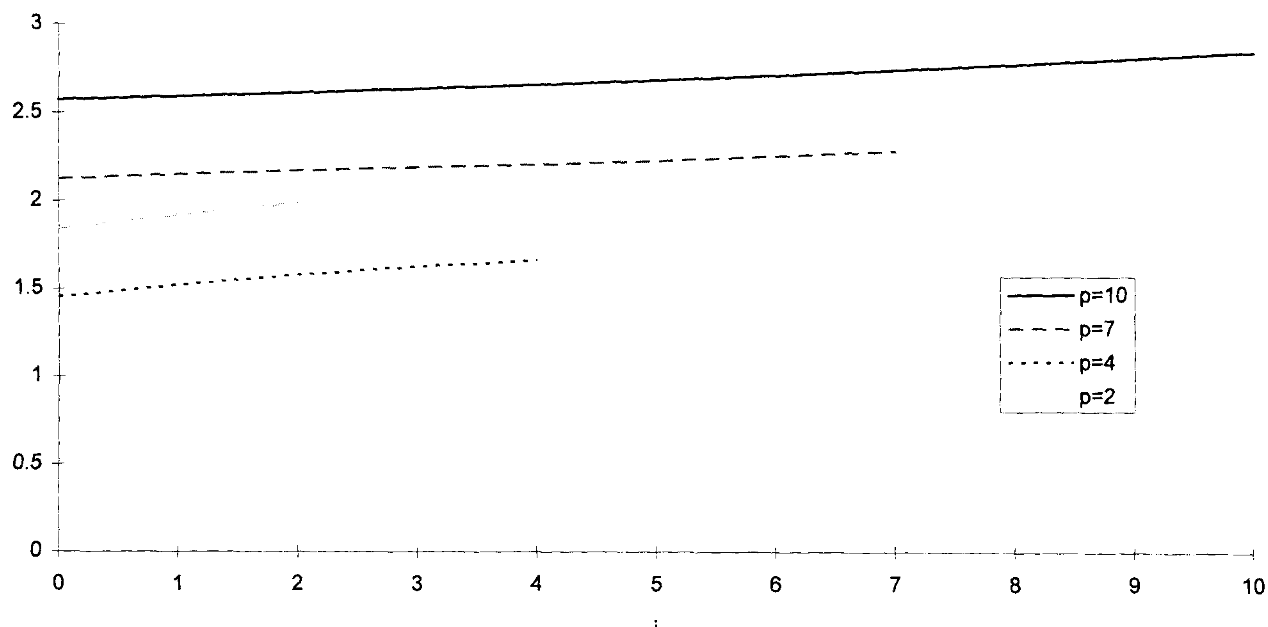

Figure 10. Behavior of r_1, $i = 0,\ldots,p$ for various p, Example 1.

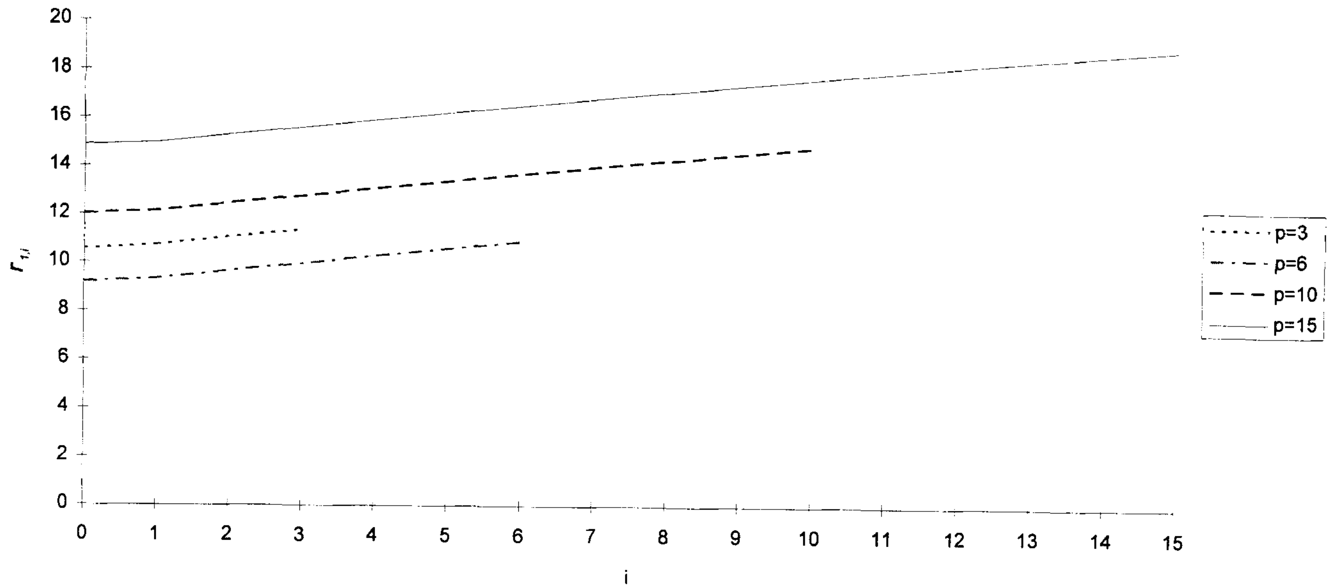

Figure 11. Plot of $r_{1,i}$, $i=0,\ldots,p$ for various p, Example 2.

The multivariable plant considered, we see the same trend for P vs. p (Figure 14), though not as pronounced as in the SISO example. Here too, we find that for the case with no modeling error, a larger p is better. However, for cases with even a little modeling error, a value of p between 3 and 9 seems best. We also observe that the P vs. p curve is jagged, which we cannot fully explain at this time.

Conclusions

In this article we have presented robust stability conditions and have demonstrated the robust performance features of multivariable EQDMC systems. Based on these conditions, we have proposed guidelines for tuning closed-loop EQDMC systems so as to guarantee closed-loop stability and good performance. Through simulations, we have shown that the end condition proposed here indeed has a stabilizing effect on an otherwise unstable constrained DMC closed-loop system. The approach presented in this article can be extended to the various forms of multivariable MPC that use a pulse or step response process model, such as model predictive heuristic control (MPHC) (Richalet et al., 1978), model algorithmic control (MAC) (Mehra et al., 1979), dynamic matrix control (DMC) (Cutler and Ramaker, 1980; Prett and Gillette, 1979), LDMC (Morshedi et al., 1985), QDMC (Garcia and Morshedi, 1986), provided uncertainty is suitably quantified as

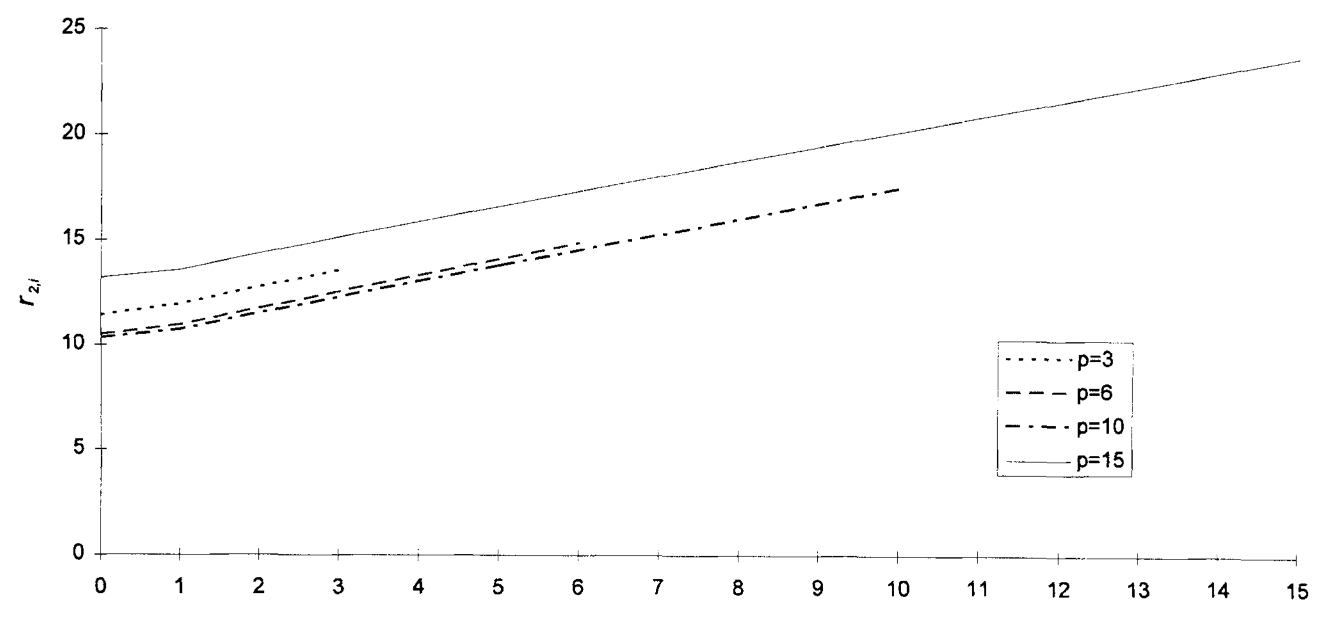

Figure 12. Plot of $r_{2,i}$, $i=0,\ldots,p$ for various p, Example 2.

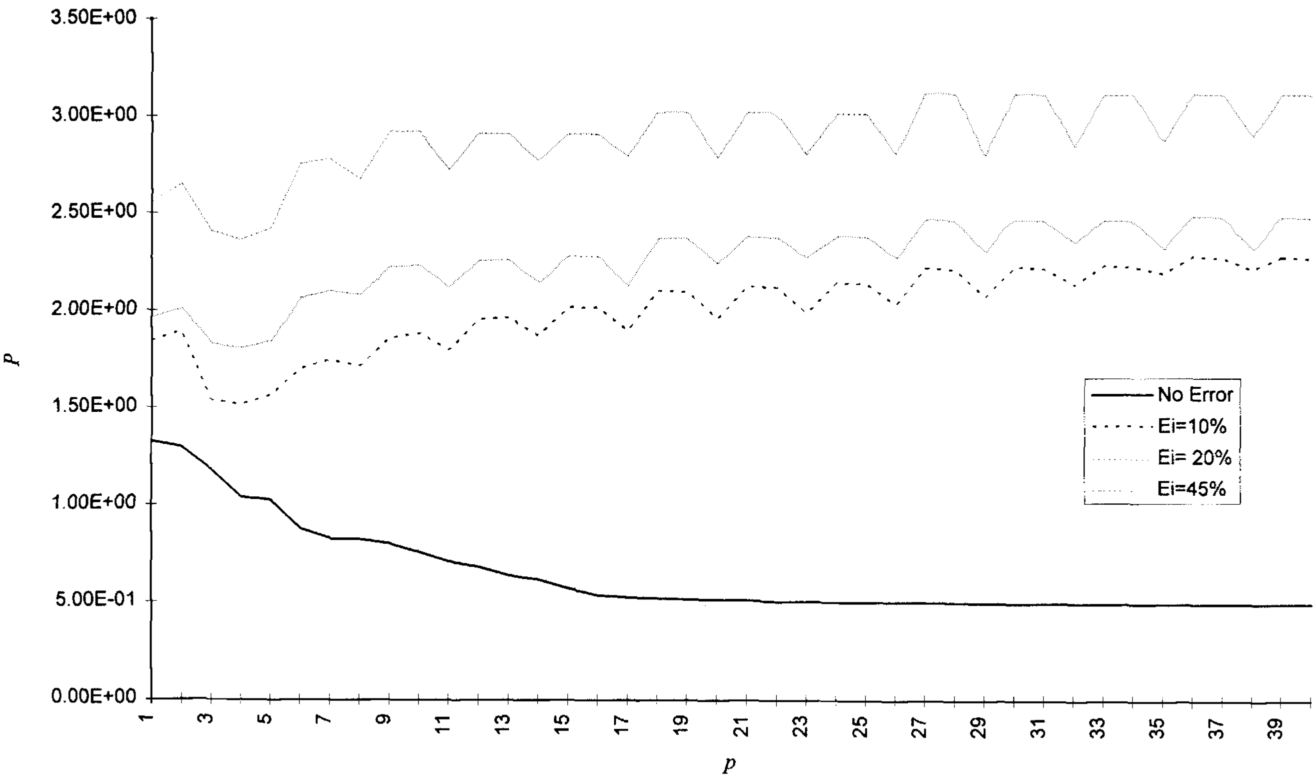

Figure 13. Dependence of the performance index, *P*, on the control horizon, *p*, for Example 1.

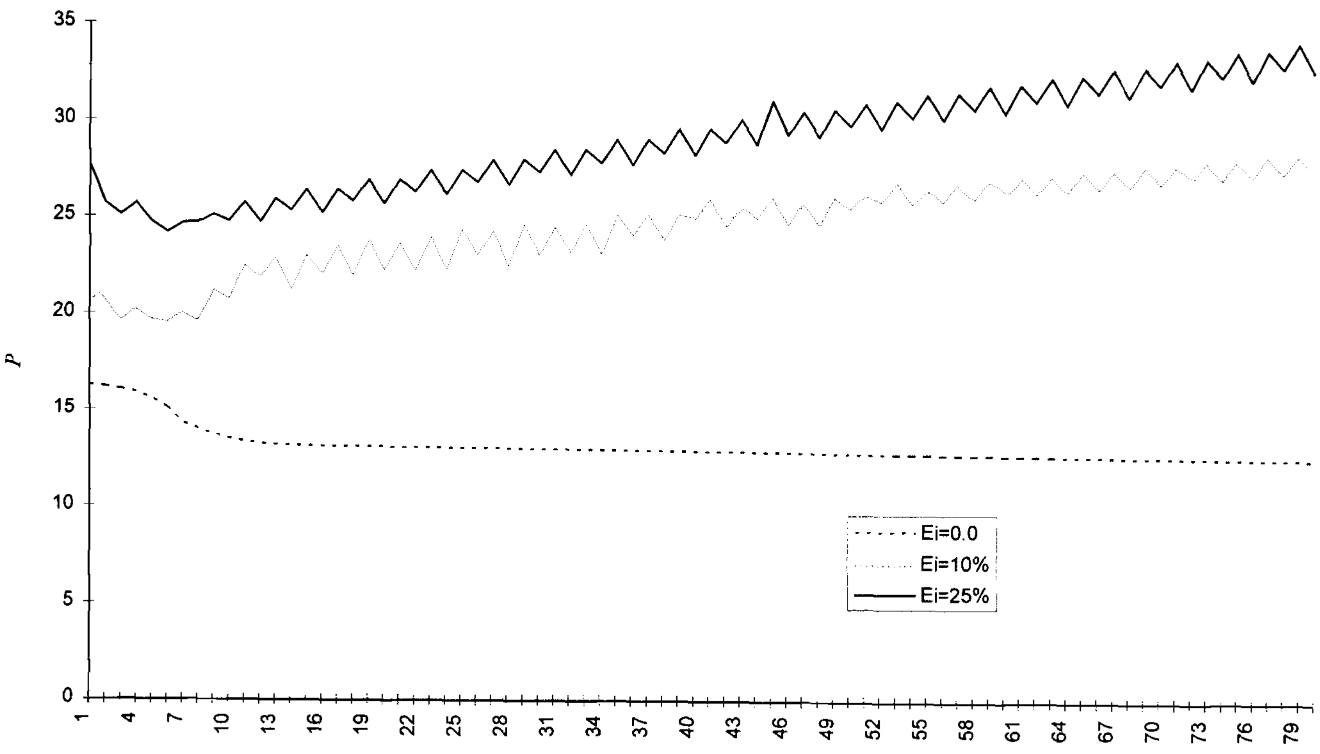

Figure 14. Dependence of the performance index, *P*, on the control horizon, *p*, for Example 2.

perturbations on the model's coefficients. The end condition can be naturally incorporated in all of the preceding MPC variations, and stability and performance issues can be subsequently investigated. It should be stressed that the robust stability conditions in this work are only sufficient. Although our experience with a few applications has shown that these conditions may not be overly conservative, additional work is needed. Both necessary and sufficient conditions would be the ultimate target.

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